Introduction to Representation Theory
Mid-Terminal Examination

February 26 2016

This exam is of 40 marks. Please read all the questions carefully and do not cheat. You are allowed to use *J-P Serre - Linear Representations of Finite Groups*. Good luck! (40)

Unless otherwise stated V will be a finite dimensional vector space over $\mathbb{C}$.

1. Let G_1 and G_2 be two groups and (V_1, ρ_1), (V_2, ρ_2) be two *irreducible* representations of G_1 and G_2 respectively.

   1. Show that $(V_1 \otimes V_2, \rho_1 \otimes \rho_2)$ is an *irreducible representation* of $G_1 \times G_2$. (5)

   2. Show that if $G := G_1 = G_2$ and

   $$\Delta : G \to G \times G$$

   $$\Delta(g) = (g, g)$$

   is the diagonal embedding then for some irreducible representations V_1 and V_2 of G the tensor product $V_1 \otimes V_2$ is *not necessarily* an irreducible representation of the subgroup $\Delta(G) \subset G \times G$. (5)

2. Let ψ be a character of a representation of G such that $\psi(g) = 0$ for all $g \neq 1$ in G. Show that $\psi = n r_G$ for some $n \in \mathbb{Z}$, where r_G is character of the regular representation. (5)

3. Let C_n denote the cyclic group of order n. Let $G = G_{21} := C_7 \rtimes C_3$, be the *semi-direct product* of C_7 and C_3 determined by the following relations: If $C_3 = \langle a \rangle$ and $C_7 = \langle b \rangle$, then one has

- Any element of G is of the form $a^m b^n$ for $0 \leqslant m < 3$, $0 \leqslant n < 7$.
- $a^3 = 1$, $b^7 = 1$
- $aba^{-1} = b^2$

Answer the following questions about G:

1. Write down a set of representatives for the the conjugacy classes of G. (5)

2. How many irreducible representations does G have? (2)

3. Let ψ_r, $0 \leqslant r < 7$ denote the 1 dimensional irreducible representation of C_7 given by

$$\psi_r(b^k) = e^{\frac{2\pi i r k}{7}}$$

   Let $\rho_r = \mathrm{Ind}_{C_7}^{G_{21}}(\psi_1)$ be the induced representation. Write down the matrix corresponding to $\rho_r(ab)$. (5)

4. For what values of r is ρ_r irreducible? (3)

5. Write down the character table of G. (10)